Game Theory
Everyone chooses a real number between 0 and 100. The winning number is the one closest to two-thirds of the average of the numbers chosen.

Everyone choosing the winning number will split the prize.
A Game

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Try It!
Reasoning about the Game

The largest number you can choose is 100.

So the right answer will be less than 67.
Reasoning about the Game

The largest number you can choose is 100.

So the right answer will be less than 67.

It doesn’t make sense to choose a number larger than 67 (if your goal is to win).
Reasoning about the Game II

But...everyone must make this computation...
Reasoning about the Game II

But...everyone must make this computation...

So really no one will choose a number larger than 67.
Reasoning about the Game II

But...everyone must make this computation...

So really no one will choose a number larger than 67.

And the winning number won’t be larger than

\[ \left( \frac{2}{3} \right)^2 \cdot 100 \approx 44 \]
Reasoning about the Game III

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Reasoning about the Game III

But...everyone must make this computation...

So really no one will choose a number larger than

\[
\left(\frac{2}{3}\right)^2 \cdot 100 \approx 44
\]

And the winning number won’t be larger than

\[
\left(\frac{2}{3}\right)^3 \cdot 100 \leq 30
\]
Reasoning about the Game IV

But...everyone must make this computation...
Reasoning about the Game IV

But...everyone must make this computation...

Where will it all end? How many rounds of reasoning do you expect people to go through?
Reasoning about the Game IV

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Where will it all end? How many rounds of reasoning do you expect people to go through?

If all the participants are rational, and know that they all are rational we would expect everyone to say 0.
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If all the participants are rational, and know that they all are rational we would expect everyone to say 0.

Playing in practice it is rare to see more than 3 iterations even in specialized populations---PhD economists, Fortune 500 CEO’s, Caltech undergrads...
Equilibrium

Suppose that everyone chose the number 0.
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Then even if the answers of everyone else were revealed to you, and you were allowed to choose again, you would have no reason to behave differently!
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A set of strategies where every player satisfies this property is known as a Nash Equilibrium.
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Everyone choosing 0 is the unique Nash Equilibrium for the 2/3 of average game.
Another Example: RPS

Classic game of Rock, Paper, Scissors
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Classic game of Rock, Paper, Scissors
Rock, Paper, Scissors

We can represent the outcomes of a two player game by a matrix.

\[
\begin{array}{ccc}
0,0 & -1,1 & 1,-1 \\
1,-1 & 0,0 & -1,1 \\
-1,1 & 1,-1 & 0,0 \\
\end{array}
\]

Image from Costis Daskalakis
Rock, Paper, Scissors

What is the Nash Equilibrium in this game?

<table>
<thead>
<tr>
<th></th>
<th>0,0</th>
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<th>1,-1</th>
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</tr>
<tr>
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</table>
Strategies

A strategy is a choice of action in a game---rock, paper, or scissors in this case.
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Randomized Strategies

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Randomized Strategies

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Alice puts a probability distribution $p$ over her possible actions, and plays action $i$ with $p(i)$.

Bob puts a probability distribution over his possible actions.

Now we look at the expected payoff.
Expected Payoff

Alice’s expected payoff:

\[ \text{pr(Alice Rock)} \text{pr(Bob Scissors)} - \text{pr(Alice Rock)} \text{pr(Bob Paper)} \]

\[ \text{pr(Alice Paper)} \text{pr(Bob Rock)} - \text{pr(Alice Paper)} \text{pr(Bob Rock)} \]

\[ \text{pr(Alice Scissors)} \text{pr(Bob Paper)} - \text{pr(Alice Scissors)} \text{pr(Bob Rock)} \]
Expected Payoff

Alice’s expected payoff:

$$\text{pr}(\text{Alice Rock}) \text{pr}(\text{Bob Scissors}) - \text{pr}(\text{Alice Rock}) \text{pr}(\text{Bob Paper})$$

$$\text{pr}(\text{Alice Paper}) \text{pr}(\text{Bob Rock}) - \text{pr}(\text{Alice Paper}) \text{pr}(\text{Bob Rock})$$

$$\text{pr}(\text{Alice Scissors}) \text{pr}(\text{Bob Paper}) - \text{pr}(\text{Alice Scissors}) \text{pr}(\text{Bob Rock})$$

If they play rock, paper, scissors for a long time with these strategies, the amount of money won by Alice will converge to this value.
Expected Payoff

\[ pr(\text{Alice Rock})[pr(\text{Bob Scissors})-pr(\text{Bob Paper})] + \]
\[ pr(\text{Alice Paper})[pr(\text{Bob Rock})-pr(\text{Bob Scissors})] + \]
\[ pr(\text{Alice Scissors})[pr(\text{Bob Paper})-pr(\text{Bob Rock})] \]
Expected Payoff

\[ \text{pr}(\text{Alice Rock})[\text{pr}(\text{Bob Scissors})-\text{pr}(\text{Bob Paper})] + \]
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\[ \text{pr}(\text{Alice Scissors})[\text{pr}(\text{Bob Paper})-\text{pr}(\text{Bob Rock})] \]

Say Alice knows Bob’s strategy. What is her best response?
Rock, Paper, Scissors

The Nash Equilibrium is for Alice and Bob to play uniformly at random.
Role of Mixed Strategies

Notice that at the Nash Equilibrium, Alice also had a pure strategy best response.

The usefulness of mixed strategies is not about maximizing payoff...

It is about providing countering strategies to achieve equilibrium.
History

Rock, Paper, Scissors is an example of a zero-sum game: whatever I win, you lose.
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Both of these proofs were non-constructive, using something called Brouwer’s fixed point theorem.
Brouwer Fixed Point Theorem: First Case

Any continuous function from $[0,1]$ to itself has a fixed point.
Brouwer Fixed Point Theorem: First Case

Note: Theorem is not true if take the interval $0 < x < 1$. 
Brouwer Fixed Point
Theorem: Second Case

Any continuous function from $[0,1] \times [0,1]$ to itself has a fixed point.
Any continuous function from $[0,1] \times [0,1]$ to itself has a fixed point.
Brouwer Fixed Point Theorem: Second Case

Any continuous function from \([0,1] \times [0,1]\) to itself has a fixed point.

Not a disk
Brouwer Fixed Point
Theorem: Second Case

Any continuous function from \([0,1] \times [0,1]\) to itself has a fixed point.

- Not a disk
- Not Bounded
Brouwer Fixed Point Theorem: Second Case

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Brouwer Fixed Point Theorem: Second Case

Any continuous function from \([0,1] \times [0,1]\) to itself has a fixed point.
Example: Penalty Shot Game

Strategies in the Penalty Shot game can be represented by \((p, q) \in [0, 1] \times [0, 1]\). Here \(p\) is probability Alice goes right, and \(q\) is probability Bob goes right.
Nash’s Proof Idea

We define a function from \([0,1] \times [0,1]\) to itself where fixed points correspond to Nash Equilibria.

We map a strategy \((p,q)\) to the strategy \((p',q')\) where \(p'\) is Alice’s best response to Bob playing \(q\), and \(q'\) is Bob’s best response to Alice playing \(p\).

A fixed point of this function will be a Nash equilibrium!
Visualizing Nash’s Construction

<table>
<thead>
<tr>
<th>Kick</th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-1, 1</td>
</tr>
<tr>
<td>Left</td>
<td>-1, 1</td>
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Penalty Shot Game

Slide from Costis Daskalakis
But how hard are they to find?

“If your laptop can’t find the equilibrium, then how can the market?”

Kamal Jain, Microsoft Research
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The practical relevance of the notion of equilibrium is tied to the question of its computational difficulty.
Zero-Sum Games

For two-player zero-sum games we can find equilibrium strategies efficiently.

This can be done by linear programming.
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von Neumann meets Dantzig
Linear programming

Can Alice achieve a payoff of $c$?
Linear programming

Can Alice achieve a payoff of c?

We know that the best counter of Bob can be a pure strategy. Alice needs to come up with a strategy $p_{\text{rock}}, p_{\text{paper}}, p_{\text{scissors}}$ such that every pure counter of Bob still gives her payoff at least c.
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System of inequalities: $p_{\text{rock}} + p_{\text{paper}} + p_{\text{scissors}} = 1$
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Payoff against Bob’s rock: \( p_{\text{paper}} - p_{\text{scissors}} \geq c \)
Linear programming

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Payoff against Bob’s rock: $p_{\text{paper}} - p_{\text{scissors}} \geq c$
What about general games?

No efficient algorithm is known to find a Nash Equilibrium in a general two-player game.

It is thought unlikely to be NP-complete.

An exciting line of recent research has shown that finding Nash equilibria is equivalent to finding Brouwer fixed points.