Problems

1. [2.13 DHS, Required for Grading] In many pattern recognition problems one has the option either to assign the pattern to one of \( c \) classes, or to reject it as being unrecognizable. If the cost for rejects is not too high, rejection may be a desirable action. Let

\[
\lambda_{(i|\omega_j)} = \begin{cases} 
0 & i = j \quad i, j = 1, \ldots, c \\
\lambda_r & i = c + 1 \\
\lambda_s & \text{otherwise,}
\end{cases}
\]

where \( \lambda_r \) is the loss incurred for choosing the \((c + 1)\)th action, rejection, and \( \lambda_s \) is the loss incurred for making any substitution error. Show that the minimum risk is obtained if we decide \( !i \) if

\[
\frac{p(x|!i)p(!i)}{\sum_{j=1}^{c+1} p(x|!j)p(!j)} > 1
\]

for all \( j \) and if

\[
\frac{p(x|!i)p(!i)}{\sum_{j=1}^{c+1} p(x|!j)p(!j)} = 1 \quad \Rightarrow \quad r = s
\]

otherwise.

What happens if \( r = 0 \)? What happens if \( r > s \)?

2. [2.14 DHS, Required for Grading] Consider the classification problem with rejection option.

(a) Use the results of Problem 2.13 above to show that the following discriminant functions are optimal for such problems:

\[
g_i(x) = \begin{cases} 
p(x|!i)p(!i) & i = 1, \ldots, c \\
\frac{\lambda_i - \lambda_r}{\lambda_s} \sum_{j=1}^{c+1} p(x|!j)p(!j) & i = c + 1.
\end{cases}
\]

(b) Plot these discriminant functions and the decision regions for the two category one dimensional case having

- \( p(x|!1) \sim N(1, 1) \),
- \( p(x|!2) \sim N(-1, 1) \),
- \( P(!1) = P(!2) = 1/2 \), and
- \( \lambda_r/\lambda_s = 1/4 \).

(c) Describe qualitatively what happens as \( \lambda_r/\lambda_s \) is increased from 0 to 1.

(d) Repeat for the case having

- \( p(x|!1) \sim N(1, 1) \),
- \( p(x|!2) \sim N(0, 1/4) \),
- \( P(!1) = 1/3, P(!2) = 2/3 \), and
- \( \lambda_r/\lambda_s = 1/2 \).

3. [2.24 DHS, Optional] Consider the multivariate normal density with mean \( \mu \), \( \sigma_{ij} = 0 \) and \( \sigma_{ii} = \sigma_i^2 \), that is, the covariance matrix is diagonal: \( \Sigma = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_d^2) \).

(a) Show that the evidence is

\[
p(x) = \frac{1}{\prod_{i=1}^{d} \sqrt{2\pi \sigma_i}} \exp \left[ -\frac{1}{2} \sum_{i=1}^{d} \left( \frac{x_i - \mu_i}{\sigma_i} \right)^2 \right].
\]
(b) Plot and describe the contours of constant density.

(c) Write an expression for the Mahalanobis distance from \( x \) to \( \mu \).

4. [2.32 DHS, Required for Grading] Let \( p(x|\omega_i) \sim N(\mu_i, \sigma^2 I) \) for a two-category \( d \)-dimensional problem with \( P(\omega_1) = P(\omega_2) = 1/2 \).

(a) Show that the minimum probability of error is given by

\[
P_e = \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-u^2/2} du,
\]

where \( a = \|\mu_2 - \mu_2\|/(2\sigma) \).

(b) Let \( \mu_1 = 0 \) and \( \mu_2 = (\mu_1, \ldots, \mu_d)^t \neq 0 \). Use the inequality from Problem 31 to show that \( P_e \) approaches zero as the dimension \( d \) approaches infinity.

(c) Express the meaning of this result in words.

References


Inequality in Problem 31 is:

\[
P_e = \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-u^2/2} du \leq \frac{1}{\sqrt{2\pi a}} e^{-a^2/2}
\]